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HOWARD J. WEINER

TECHNICAL REPORT NO. 25

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DEPARTMENT OF STATISTICS

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by

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I. INTRODUCTION

On a line segment $(0, a+\alpha)$, cars or segments of length α are to be parked in this manner. Choose the point X uniformly on $(0, a)$. Then the first car is parked at $(X, X+\alpha)$. The succeeding cars are placed I.I.D. as the first, except that if there is overlap with a previously-parked car, the new car is discarded, otherwise it is parked. The process is continued until no further cars can be parked. This is Model I for parking one-dimensional cars, due to Renyi (see [2], [3]).

Denote

$$(1.1) \quad M_{\alpha}(a+\alpha) = \text{mean total number of cars of length } \alpha \text{ parked in} \\ \text{accord with Model I.}$$

Then (see [2], pp. 131-132) by total probability,

$$(1.2) \quad M_{\alpha}(a+\alpha) = 1 + \frac{2}{a} \int_0^a M_{\alpha}(x) dx, \quad M_{\alpha}(x) = 0, \quad 0 \leq x < \alpha.$$

It is known that ([2], pp. 123-124)

$$(1.3) \quad \lim_{a \rightarrow \infty} \alpha a^{-1} M_{\alpha}(a) = \eta \sim .75.$$

In Model II of Solomon ([2], pp. 129-132), a one-dimensional car has coordinates $(X, X+\alpha)$, where X is uniform on $(-\alpha, a)$. The total parking boundary is the segment $(0, a+\alpha)$, for $a \geq \alpha$. If $-\alpha < X < 0$, the car is parked at $(0, \alpha)$. If $a-\alpha < X < a$, the car is parked at $(a-2\alpha, a-\alpha)$. If $0 < X < a-\alpha$, the car is parked at $(X, X+\alpha)$. A second car of length α is placed I.I.D. as the first car and parked if it does not overlap the first car. If it does overlap the first parked car, and the first parked car has coordinates $(x, x+\alpha)$, $x > \alpha$, and the second car is initially placed at $(Y, Y+\alpha)$, with $\alpha < x < Y+\alpha < x+\alpha$, the second car is parked at $(x-\alpha, x)$. If $x < Y < x+\alpha$, the second car is parked at $(x+\alpha, x+2\alpha)$ if $x+2\alpha < a$. In all other cases the second car is discarded. The process is continued as above, except that a newly placed car is also discarded if it initially overlaps another parked car, and upon moving adjacent to this car, either will not fit on the $(0, a)$ segment, or overlaps still another parked car. The process is continued until no further cars may be parked.

Denote

$$(1.4) \quad R_\alpha(a) \equiv \text{mean total number of parked cars on } (0, a) \text{ in accord with Model II.}$$

By total probability, ([2], pp. 131-132),

$$(1.5) \quad R_\alpha(a+\alpha) = 1 + \frac{2\alpha}{a+2\alpha} R_\alpha(a) + \frac{2}{a+2\alpha} \int_0^a R_\alpha(x) dx.$$

For Model II ([2], pp. 131-132),

$$(1.6) \quad \lim_{a \rightarrow \infty} \alpha a^{-1} R_\alpha(a) = \delta \sim .81.$$

It is the purpose of this paper to show that the Palasti conjecture for sequential random packing holds in the plane for two-dimensional versions

of Models I and II. This will be formulated precisely below, and in the special case of unit square size cars parked parallel to the sides of a rectangular boundary, the Palasti conjecture states that as the boundary area increases, the limiting ratio of mean total number of cars parked divided by the boundary area approaches η^2 for Model I and δ^2 for Model II. The extension of the Palasti conjecture to n-dimensions for Models I and II is given. A random car size model in one dimension for Models I and II are considered, and asymptotic results indicated.

II. MODEL I IN TWO DIMENSIONS

For Model I in two dimensions, consider the rectangular boundary with corners at $(0,0)$, $(0,b)$, $(a,0)$, (a,b) . The first car is parked in the space given by the corners (X,Y) , $(X,Y+\beta)$, $(X+\alpha,Y)$, $(X+\alpha,Y+\beta)$ where (X,Y) is chosen uniformly at random in the subrectangle $(0,0)$, $(0,b-\beta)$, $(a-\alpha,0)$, $(a-\alpha,b-\beta)$. Succeeding cars of the same size and orientation are placed I.I.D. as the first, and parked if there is no overlap with a car already parked, and otherwise discarded.

Define

(2.1) $M(a,b) \equiv$ mean total number of $\alpha \times \beta$ size parked cars in the $a \times b$ rectangle.

Lemma 1. Let $P(x,y)$ be defined for $x,y \geq 0$ and satisfy

- (2.2) (i) $P(x,y) = 0$ if either $x \leq \alpha$ or $y \leq \beta$ where α, β are positive constants.
- (ii) $xyP(x,y) \begin{matrix} \geq \\ (\leq) \end{matrix} A \int_0^{x-\alpha} ds \int_0^{y-\beta} dt P(x,t)$ where $A > 0$ is a constant and $x > \alpha, y > \beta$.

Then for $x \geq \alpha$, $y \geq \beta$,

$$(2.3) \quad P(x,y) \underset{(\leq)}{\overset{\geq}{}} 0.$$

Proof. By (2.2)(i) substituted into the right side of (2.2)(ii) with

$$(2.2) \quad (ii) \quad xyP(x,y) \geq A \int_0^{x-\alpha} ds \int_0^{y-\beta} dt P(x,t),$$

it follows that

$$(2.3) \quad P(x,y) \geq 0 \quad \text{for} \quad 2\alpha \geq x \geq \alpha, \quad 2\beta \geq y \geq \beta.$$

Iterating by substituting (2.3) into the right side of (2.2)(ii) just above proves that

$$(2.4) \quad P(x,y) \geq 0 \quad \text{for} \quad 3\alpha \geq x \geq \alpha, \quad 3\beta \geq y \geq \beta.$$

Iterating (2.4) in this manner proves the result. The other inequality is similar.

Consider the $a \times b$ rectangle with coordinates $(0,0)$, $(0,b)$, $(a,0)$, (a,b) , and rectangular $\alpha \times \beta$ cars, $\alpha, \beta \ll a, b$. Let ℓ denote the line segment $(0, b-\beta)$ to $(a, b-\beta)$. A key lemma is the following.

Lemma 2. The $\alpha \times \beta$ cars parked in the $a \times b$ rectangle according to Model I intersect line segment ℓ in segments (of length α) in accord with a one-dimensional law of Model I for cars of length α parked on a segment of length a .

Proof. The line segment ℓ must be intersected by parked cars such that no other cars can fit. Otherwise another car could be parked on ℓ . Given that this is the case, and that the x,y-coordinates which determine the placement of a car to be parked are chosen I.I.D. uniformly, then the horizontal placement and parking of cars on ℓ is independent of all other parked cars and depends only on the x-coordinate. This suffices for the proof.

Lemma 3. In Model I, for $a \geq 2\alpha$ or $b \geq 2\beta$, and $\alpha \times \beta$ cars,

$$(2.5a) \quad M(a, b+\beta) \geq M(a, b)$$

$$(2.5b) \quad \frac{M(a, b+\beta)}{a(b+\beta)} \leq \frac{M(a, b)}{ab}$$

$$(2.6a) \quad M(a, b) + M_\alpha(a) \geq M(a, b+\beta)$$

$$(2.6b) \quad M(a, b) + M_\alpha(a) \leq M(a, b+2\beta).$$

Proof. By an induction and taking derivatives of each of $a^{-1}M_\alpha(a)$, $b^{-1}M_\beta(b)$, $M_\alpha(a)$, $M_\beta(b)$ and checking their sign, it may be concluded that for $a > 2\alpha$, $b > 2\beta$,

$$(2.7a) \quad a^{-1}M_\alpha(a) \text{ and } b^{-1}M_\beta(b) \text{ are monotone decreasing, and}$$

$$(2.7b) \quad M_\alpha(a) \text{ and } M_\beta(b) \text{ are monotone increasing.}$$

Denote the $\alpha \times \beta$ cars parked in the $a \times b$ rectangle according to Model I and which intersect line ℓ of lemma 2 by row 1 of parked cars. Below row 1, the immediately adjacent cars form row 2 from one end of the rectangle to the other, and so on, until rows are exhausted, and "partial rows" form. From the independence of the (x,y) coordinates, density and monotonicity relations per row

of (2.7a), (2.7b) and a consideration of each row and partial row yields the density and monotonicity relations respectively

$$(2.8a) \quad \frac{M(a,b)}{ab} \leq \frac{M(c,d)}{cd} \quad \text{for } a \geq c, b \geq d$$

and

$$(2.8b) \quad M(a,b) \geq M(c,d) \quad \text{for } a \geq c, b \geq d.$$

Relations (2.5a) and (2.5b) are special cases of (2.8b), (2.8a) respectively. Relations (2.6a), (2.6b) follow from (2.7a), (2.8b), lemma 2, and consideration of row formation as in the above paragraph.

Lemma 4. Consider the $(a+\alpha) \times (b+\beta)$ rectangle as in Figure 2 and lemma 2. The cross-hatched subrectangle is an $\alpha \times \beta$ car, considered to be the first car parked in the rectangle.

Then for $a \geq 2\alpha$,
 $b \geq 2\beta$ and a, b multiples
of α, β respectively,

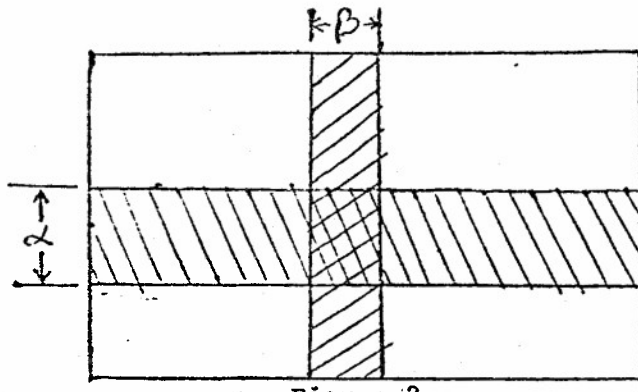


Figure 2

$$(2.9) \quad M(a+\alpha, b+\beta) \geq M_\alpha(a-\alpha) + M_\beta(b-\beta) - 1 + \frac{4}{(a-2\alpha)(b-2\beta)} \int_0^{a-2\alpha} ds \int_0^{b-2\beta} dt M(s,t),$$

$$(2.10) \quad M(a+\alpha, b+\beta) \leq M_\alpha(a+\alpha) + M_\beta(b+\beta) - 1 + \frac{4}{ab} \int_0^a ds \int_0^b dt M(s,t).$$

Proof. The inequalities are direct consequences of Lemmas 2 and 3, upon consideration of shaded areas to be replaced by one-dimensional lines of cars. Then (2.6a) establishes (2.10) and (2.6b) establishes (2.9).

Theorem 1. For Model I of Renyi in the plane,

$$(2.11) \quad \lim_{a,b \rightarrow \infty} \alpha\beta(ab)^{-1} M(a,b) = \eta^2.$$

Proof. A straightforward computation using (1.2) shows that for r an integer, positive or negative,

$$(2.12) \quad M_\alpha(a+r\alpha)M_\beta(b+r\beta) = M_\alpha(a+r\alpha) + M_\beta(b+r\beta) - 1 \\ + \frac{4}{(a+(r-1)\alpha)(b+(r-1)\beta)} \int_0^{a+(r-1)\alpha} ds \int_0^{b+(r-1)\beta} dt M_\alpha(s)M_\beta(t).$$

Now subtracting (2.12) with $r = -1$ from (2.9) and subtracting (1.12) with $r = +1$ from (2.10) yields for

$$(2.13) \quad M_\alpha(a-\alpha)M_\beta(b-\beta) \leq M(a+\alpha, b+\beta) \leq M_\alpha(a+\alpha)M_\beta(b+\beta).$$

Dividing (2.13) by ab and using (1.3) yields the result.

III. MODEL II IN TWO DIMENSIONS

Model II for sequential random packing in the plane is an extension of the one-dimensional packing model of Solomon ([2], pp. 129, 131-132) and is defined as follows. Again there is an $a \times b$ rectangular boundary and $\alpha \times \beta$ size cars to be parked with side α parallel to a and β to b . The $a \times b$ rectangular boundary has lower left corner at $(0,0)$, upper right at (a,b) . The first car may be placed with its lower left corner uniform on the rectangle $(-\alpha, -\beta)$, $(-\alpha, b)$, $(-\beta, a)$, (a, b) .

There are two cases. First, if the lower left corner of the first car lands in the strip $(-\alpha, 0)$, $(0, 0)$, $(-\alpha, b-\beta)$, $(0, b-\beta)$, the car is shifted horizontally to the right until its lower left corner is on the vertical axis, and the car is parked there. Similarly, if the car is placed with lower left corner in the other strips so that part of the car is outside of the $a \times b$ boundary, the car is moved vertically or horizontally until it "fits" into the boundary. If the lower left corner is initially in the strip $(-\alpha, -\beta)$, $(-\alpha, 0)$, $(0, -\beta)$, $(0, 0)$, the car is moved up and parked with its lower left corner at $(0, 0)$. Similarly with the other 3 locations at the corners of the $a \times b$ rectangle. For the second case, if the first car falls within the $a \times b$ rectangle, it is parked there. A second car is parked I.I.D. as the first, except that if the initial placement overlaps that of the already parked first car, the second car is moved to the left or down or diagonally if its lower left corner is not within the first car, and to the right or up (or diagonally if necessary) if it is. If the second car still cannot fit, it is discarded. Similarly if there are at least two already parked cars the next car is parked I.I.D. as the others, except that it is discarded if it cannot fit into the $a \times b$ boundary or overlaps a second parked car after it is maneuvered as indicated. The process continues until no further cars may be parked.

Denote

$$(3.1) \quad R_{\alpha, \beta}(a, b) \equiv R(a, b) = \text{mean total number of } \alpha \times \beta \text{ size cars} \\ \text{parked on an } a \times b \text{ rectangle according to the Model II.}$$

Lemma 5. Let $R(x, y)$ be defined for $x \geq 0$, $y \geq 0$, and satisfy

$$(3.2) \quad (i) \quad R(x,y) = 0 \quad \text{if } x \leq \alpha, y \leq \beta,$$

$$(3.2) \quad (ii) \quad R(x,y) \begin{matrix} \geq \\ (\leq) \end{matrix} AR(x-\alpha,y) + BR(x,y-\beta) \\ + CR(x-\alpha,y-\beta) \\ + D \int_0^{x-\alpha} ds \int_0^{y-\beta} dt R(s,t),$$

for positive constants A, B, C, D. Then for $x \geq \alpha, y \geq \beta$,

$$(3.3) \quad R(x,y) \begin{matrix} \geq \\ (\leq) \end{matrix} 0.$$

Proof. The proof is similar to that of Lemma 1 and is omitted.

Lemma 6. Consider the $a \times b$ rectangular boundary with coordinates $(0,0)$, $(0,b)$, $(a,0)$, (a,b) and the line segment from $(0,b-\beta)$ to $(a,b-\beta)$. Cars of size $\alpha \times \beta$ are to be parked on the rectangular boundary $a \times b$ above with α -side parallel to a -side, and β -side parallel to the b -side in accord with Model II in the plane until no further cars may be parked.

Then the number of $\alpha \times \beta$ parked cars that intersect line ℓ above is distributed in accord with the one-dimensional law of Model II for cars of length α parked on a segment of length a .

Proof. The argument is as in Lemma 2.

Lemma 7. For Model II in the plane, for $a \geq 2\alpha, b \geq 2\beta$

$$(3.4) \quad R(a,b+\beta) \geq R(a,b)$$

$$(3.5) \quad \frac{R(a, b+\beta)}{a(b+\beta)} \leq \frac{R(a, b)}{ab}.$$

$$(3.6) \quad R(a, b) + R_\alpha(a) \geq R(a, b+\beta),$$

$$(3.7) \quad R(a, b) + R_\alpha(a) \leq R(a, b+2\beta).$$

Proof. The relations (3.4), (3.5) follow from Lemma 6 as Lemma 3 follows from Lemma 2. Then (3.6), (3.7) follow from (3.4), (3.5) as (2.6a), (2.6b) follow from (2.5a), (2.5b).

Lemma 8. Denote, for Model II,

$$(3.8) \quad T(a, b) \equiv R_\alpha(a)R_\beta(b).$$

Then for k some integer, positive or negative,

$$(3.9) \quad T(a+k\alpha, b+k\beta) = R_\alpha(a+k\alpha) + R_\beta(b+k\beta) - 1 + \frac{2\alpha}{a+(k+1)\alpha} T(a+(k-1)\alpha, b+k\beta) \\ - \frac{2\alpha R_\alpha(a+(k-1)\alpha)}{a+(k+1)\alpha} + \frac{2\beta T(a+k\alpha, b+(k-1)\beta)}{b+(k+1)\beta} - \frac{2\beta R_\beta(b+(k-1)\beta)}{b+(k+1)\beta} \\ - \frac{4\alpha\beta T(a+(k-1)\alpha, b+(k-1)\beta)}{(a+(k+1)\alpha)(b+(k+1)\beta)} + \frac{4}{(a+(k+1)\alpha)(b+(k+1)\beta)} \int_0^{a+(k-1)\alpha} ds \int_0^{b+(k-1)\beta} dt T(s, t).$$

Proof. This is a straightforward computation using (1.5).

Lemma 9. For Model II in the plane, and $a \geq 2\alpha$, $b \geq 2\beta$.

$$\begin{aligned}
(3.10) \quad R(a+\alpha, b+\beta) &\leq R_\alpha(a+\alpha) + R_\beta(b+\beta) - 1 + \frac{2\alpha}{a+2\alpha} R(a, b+\beta) - \frac{2\alpha}{a+2\alpha} R_\alpha(a) \\
&+ \frac{2\beta R(a+\alpha, b)}{b+2\beta} - \frac{2\beta}{b+2\beta} R_\beta(b) - \frac{4\alpha\beta R(a, b)}{(a+2\alpha)(b+2\beta)} \\
&+ \frac{4}{(a+2\alpha)(b+2\beta)} \int_0^a ds \int_0^b dt R(s, t),
\end{aligned}$$

$$\begin{aligned}
(3.11) \quad R(a+\alpha, b+\beta) &\geq R_\alpha(a-\alpha) + R_\beta(b-\beta) - 1 + \frac{2\alpha}{a} R(a-2\alpha, b-\beta) - \frac{2\alpha}{a} R_\alpha(a-2\alpha) \\
&+ \frac{2\beta R(a-\alpha, b-2\beta)}{b} - \frac{2\beta R_\beta(b-2\beta)}{b} - \frac{4\alpha\beta R(a-2\alpha, b-2\beta)}{ab} \\
&+ \frac{4}{ab} \int_0^{a-2\alpha} ds \int_0^{b-2\beta} dt R(s, t).
\end{aligned}$$

Proof of (3.10). Considering Figure 1 as applying to the two-dimensional version of Model II, there are two cases. In the first case, the first car falls within the rectangular boundary as in Figure 1. In this case the shaded strips are replaced with one-dimensional lines of cars and (3.6) is used to account for the terms 1, 2, 3, 9 on the right side of (3.10). Term 3, the -1 on the right of (3.10) is to avoid double-counting the car common to the two shaded strips of Figure 1. The first case part of the inequality follows from (3.6). In the second case the first car initially falls partially outside the rectangular parking boundary and is parked with one side on the rectangular boundary. If, for example, the first car is parked with its lower horizontal side on the lower horizontal boundary of the rectangular parking boundary, then by the analog of (3.6),

$$(3.12) \quad R(a+\alpha, b+\beta) \leq R(a+\alpha, b) + R_{\alpha}(a+\alpha)$$

and from a computation and induction using (1.5) or from (3.4),

$$(3.13) \quad R_{\beta}(b+\beta) \geq R_{\beta}(b).$$

Use of (3.12), (3.13) accounts for the terms 1, 2, 3, 6, 7 on the right of (3.10). The "-1", term 3 on the right of (3.10) is to avoid double counting a car common to the two perpendicular shaded strips. Considering a first car parked with a vertical side on a vertical side of the parking boundary accounts in addition for terms 4, 5 on the right side of (3.10). The term 8 on the right of (3.10) is subtracted to avoid double counting when the first car is parked in one of the four corners of the rectangular boundary. The double counting arises since a corner may be considered as both part of a horizontal and vertical shaded strip. This yields (3.10).

To obtain (3.11), (3.7) is used to obtain

$$(3.14) \quad R(a+\alpha, b+\beta) \geq R(a-2\alpha, b-\beta) + R_{\alpha}(a-\alpha)$$

and

$$(3.15) \quad R(a+\alpha, b+\beta) \geq R(a-\alpha, b-2\beta) + R_{\beta}(b-\beta).$$

Also, (3.4) or an induction based on (1.5) yields

$$(3.16) \quad R_{\alpha}(a-\alpha) \geq R_{\alpha}(a-2\alpha)$$

$$(3.17) \quad R_{\beta}(b-\beta) \geq R_{\beta}(b-2\beta).$$

Again considering the first car to be parked in one of two ways as in the above argument used to establish (3.10) and use of (3.14) - (3.17), and the double-counting arguments as used for (3.10) establish (3.11).

Theorem 2. For Model II in the plane, the Palasti conjecture holds, that is,

$$(3.18) \quad \lim_{a,b \rightarrow \infty} \alpha\beta(ab)^{-1} R(a,b) = \delta^2 \sim .65.$$

Proof. Subtracting (3.9) for $k = +1$ from (3.10) and subtracting (3.9) for $k = -1$ from (3.11) yields inequalities of the form (3.2)(ii) in both directions. Then application of Lemma 5 yields

$$(3.19) \quad R_{\alpha}(a-\alpha)R_{\beta}(b-\beta) \leq R(a+\alpha, b+\beta) \leq R_{\alpha}(a+\alpha)R_{\beta}(b+\beta).$$

Then dividing (3.19) by ab and using (1.6) yields the result of Theorem 2.

Remark. The arguments for Theorems 1 and 2 in the plane clearly carry over to higher dimensional analogs of Models I and II, respectively. The detailed descriptions of Models I and II for dimensions higher than two will be omitted, but the result will be stated.

Denote the n -vectors

$$(3.20) \quad \underline{a} = (a_1, \dots, a_n)$$

$$\underline{\alpha} = (\alpha_1, \dots, \alpha_n)$$

and

$$\pi a \equiv a_1 \cdot a_2 \cdot \dots \cdot a_n.$$

Denote

$$(3.21) \quad M(\underline{a}) = \text{mean total number of } \underline{\alpha}\text{-cars parked in an } \underline{a}\text{-rectangle according to Model I in } n\text{-dimensions.}$$

$$(3.22) \quad R(\underline{a}) = \text{mean total number of } \underline{\beta}\text{-cars parked in an } \underline{a}\text{-rectangle according to Model II in } n\text{-dimensions.}$$

Theorem 3. For Model I in $n \geq 3$ dimensions, the Palasti conjecture holds, that is,

$$(3.23) \quad \lim_{a_1, a_2, \dots, a_n \rightarrow \infty} (\pi\alpha)(\pi a)^{-1} M(\underline{a}) = \eta^n.$$

For Model II in $n \geq 3$ dimensions, the Palasti conjecture holds, that is,

$$(3.24) \quad \lim_{a_1, \dots, a_n \rightarrow \infty} (\pi\alpha)(\pi a)^{-1} R(\underline{a}) = \delta^n.$$

Proof. The argument is similar to that for two dimensions, and an induction on the result for n -dimensions is used just as the one-dimensional Palasti limit theorem is used for the Palasti limit in the plane. The details are omitted.

IV. RANDOM ~~SIZE~~ CAR SIZE

An extension in which car size is chosen from a distribution independent of the parking mechanism and I.I.D. is indicated below for the first moments in the one-dimensional cases of Models I and II. A model for random car lengths in one dimension for Model I where the car size distribution is state-dependent on the available distribution of parking space lengths was considered in [1] and the asymptotic first moment obtained.

The model here is simpler. Car sizes are chosen I.I.D. from a distribution with density f , distribution function F , with finite positive mean.

Using the same notation for Model I in Section 1, let

$$(4.1) \quad M(\underline{a}) \equiv \text{mean number of cars which can be parked in this random car size version of Model I, after averaging over car size.}$$

It follows that

$$(4.2) \quad M(a) = F(a) + \frac{2}{a} \int_0^a M(a-\xi) F(\xi) d\xi.$$

Denote

$$(4.3) \quad L(s) \equiv \int_0^\infty e^{-sx} M(x) dx$$

and

$$(4.4) \quad \varphi(s) \equiv \int_0^\infty e^{-sx} f(x) dx$$

then taking Laplace transforms of (5.2) after multiplying by a yields

$$(4.5) \quad -L_s = -\left(\frac{\varphi}{s}\right)_s + \frac{2\varphi}{s} L.$$

Assume

$$(4.6) \quad 0 < \int_0^\infty \varphi'(u) \ln(u) du < \infty.$$

This holds if, for example, the domain of F is $[a, b]$ for $0 < a < b < \infty$.

Denote

$$(4.7) \quad \mu \equiv \int_0^\infty \varphi'(u) \ln(u) du.$$

it follows that, by the previous methods,

$$(4.8) \quad L(s) = e^{2 \int_s^\infty \frac{\varphi(\xi)}{\xi} d\xi + 2\mu \int_s^\infty \left(e^{-2 \int_\xi^\infty \frac{\varphi(u)}{u} du - 2\gamma} \right) \left(-\left(\frac{\varphi(\xi)}{\xi}\right)_\xi \right) d\xi}$$

so that, using [2], p. 132

$$(4.9) \quad \lim_{s \downarrow 0} s^2 L(s) = k,$$

where

$$(4.10) \quad k = \int_0^\infty \left(e^{-2 \int_\xi^\infty \frac{\varphi(u)}{u} du - 2\mu} \right) \left(-\left(\frac{\varphi(\xi)}{\xi} \right)_\xi \right) d\xi,$$

so that for this model

$$(4.11) \quad \lim_{a \rightarrow \infty} a^{-1} M(a) = k.$$

For Model II with the same randomness in car length, denote

$$(4.12) \quad R(a) = \text{mean number of cars which may be parked in Model II in } [0, a].$$

Let

$$(4.13) \quad J(s) \equiv \int_0^\infty e^{-sx} R(x) dx.$$

Then

$$(4.14) \quad R(a) = F(a) + \frac{2}{a} \int_0^a \alpha R(a-\alpha) f(\alpha) d\alpha + \frac{2}{a} \int_0^a R(a-\xi) F(\xi) d\xi,$$

yielding, where

$$(4.15) \quad J_s \equiv \frac{d}{ds} J(s)$$

$$(4.16) \quad -J_s = -\left(\frac{\varphi}{s} \right)_s - 2\varphi_s J + \frac{2\varphi}{s} J$$

and solving for J by using an integrating factor, taking limits as $s \downarrow 0$, by the same method as before, the final result is

$$(4.17) \quad \lim_{a \rightarrow \infty} a^{-1} R(a) = \lambda,$$

where, since the integrating factor must vanish at $\xi = \infty$, it follows that

$$(4.18) \quad \lambda = \int_0^\infty e^{\left(-2 \int_\xi^\infty \frac{\varphi(u)}{u} du - 2\varphi(\xi) - 2\mu + 2\right)} \left[-\left(\frac{\varphi(\xi)}{\xi}\right)_\xi\right] d\xi.$$

Higher dimensions for this random car size model can be treated by methods of the preceding sections, then averaging over car size. This is straightforward and will not be carried out here.

One possible 2-dimensional extension is as follows, by the methods of the previous sections.

(4.19) Let $W(a,b)$ = mean total number of cars of dimensions X_i, Y_i which can be parked in an axb rectangle according to the two-dimensional Renyi model, where X_i, Y_i are I.I.D. with density f .

Let

(4.20) $U(a,b)$ = mean total number of cars of dimensions X_i, Y_i which can be parked in an axb rectangle according to the two-dimensional Solomon model, where X_i, Y_i are I.I.D. with density f .

Theorem 4

$$(4.21) \quad \lim_{a,b \rightarrow \infty} (ab)^{-1} W(a,b) = k^2.$$

$$(4.22) \quad \lim_{a,b \rightarrow \infty} (ab)^{-1} U(a,b) = \lambda^2.$$

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The Palasti conjecture on the asymptotic mean proportion of coverage is verified for the sequential random packing of rectangular cars with sides parallel to rectangular boundaries in the models of Renyi and Solomon. The extension to n -dimensions is given. An extension to a random car size model is indicated.

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